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Multifractality in time series

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Abstract. We apply the concepts of multifractal physics to financial time series in order to characterize the onset of crash for the Standard & Poor 500 (S&P500) stock index $x(t)$. It is found that within the framework of multifractality, the 'analogous' specific heat of the S&P500 discrete price index displays a shoulder to the right of the main peak for low time-lag values. For decreasing T, the presence of the shoulder is a consequence of the peaked, temporal $x(t+T)-x(t)$ fluctuations in this regime. For large time lags ($T > 80$), we have found that C_q displays typical features of a classical phase transition at a critical point. An example of such dynamic phase transition in a simple economic model system, based on a mapping with multifractality phenomena in random multiplicative processes, is also presented by applying former results obtained with a continuous probability theory for describing scaling measures.

1. Introduction

In this work we apply the concepts of multifractal physics to financial time series in order to characterize the onset of crash for the Standard & Poor 500 (S&P500) stock index. We shall present an example of dynamic phase transition in a simple economic model system based on a mapping with multifractality phenomena in random multiplicative processes and by applying former results obtained with a continuous probability theory for describing scaling measures.

An attempt is made to characterize the presence of stock market crashes by solving for a price equation from a nonlinear equilibrium model and showing how multifractal physics measures can be generated from this equation. We found that an 'analogous' specific heat C_q of the S&P500 price data displays a shoulder to the right of the main peak as a function of time lags. For large time lags, C_q resembles a classical phase transition at a critical point. We explain this dynamic phase transition by a mapping with multifractality phenomena in random multiplicative processes. Within this description the temporal price variations of a commodity displays features of an 'analogous' phase transition from inflated to devalued prices, when the excess demand is not linear in the asset price. An analytical expression for C_q of the economic model system is derived.

Finance and physics joint 'ventures' have attracted considerable interest in the literature for many years [1]. These efforts have allowed one to pursue analogies between stock market dynamics and stochastic models commonly used in the statistical physics of complex systems [2]. Such parallel analysis have been useful to best quantify and understand possible correlations in financial data by measuring the autocorrelation function [3] and the power spectrum [4]. Another notable example is the analogy with the scaling properties observed in turbulence [5–7].

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Parallels have also been drawn between a very simple theory of financial markets and the quantum gauge theory [8] (see also comments in [9]). Other interesting studies for investment strategies in diversified portfolio stocks have been recently discussed in terms of products of random matrices [10] and of multiplicative random walks [11]. Realistic price fluctuations have also been found to emerge in the adaptation of a system (i.e. the traders) to complex environments [12] and from the self-organization of a (closed) system of traders without external influences [13].

Thus, the financial market dynamics is still an open subject for physicists and economists. In particular, the basic principles governing the origin of stock market crashes are far from well understood by both communities [14], especially in regard to the problem of world-wide market dynamics. Using the renormalization group theory [14–17], it has been proposed that this cooperative phenomenon could be the result of a critical phase transition (see also [18]). Motivated by these suggestions, it is then tempting to determine the constraints needed to understand and describe analytically an analogous phase transition from a new perspective. That is, by using the characterization of multifractal singularities where an analogy with thermodynamics has already been established [19, 20].

Multifractality was initially proposed to treat turbulence and, in recent years, it has been applied successfully in many different fields ranging from model systems such as diffusion-limited aggregation to physiological data such as the heartbeat. The multifractal analysis of financial discrete sequences developed in this paper is another aspect in a relatively new topic of physics, so-called econophysics.

2. Multifractal characterization

This paper will focus on scaling laws in financial returns, particularly on the absolute values of returns and their scatter as a function of time. Similarly to other studies [21], we are not attempting to predict a price drop or rise over a specific time on the basis of past records, but we are after a new characterization of the presence of stock crashes, which is the added value of our distinctive approach. This is important to mention, since forecasting has been extensively studied in the econometrics literature by other techniques such as ARCH and multivariance models, depending, for example, on the data seasonal behaviour and previous experiences of the forecasters. Interesting work on fractional processes of these types in econometrics is very well known and these are beyond the scope of our work. It is important to mention that recent theoretical work has considered the possibility of fractal nature of the absolute value of returns on exchange rates [21].

Next, let us introduce our new characterization of a financial time series $x(t)$ based on the well known definitions used in multifractal physics. In figure 1 we show the temporal $x(t+T)-x(t)$ fluctuations of the S&P500 index (data available in [22]) for the 12-year period 2 January 1980–31 December 1992 as a function of the trading time lags $1 \leqslant T \leqslant 220$. From this figure, it can be observed that the peaked narrow fluctuations measured for $T = 1$ spread out for increasing T. For $T \to 80$, the maximum and minimun difference values of the so-called *Black Monday* crash measured in 1987 (shown as the largest straight line for $T = 1$ in the figure), become comparable to the valleys and picks differences of the relative S&P500 fluctuations over wide periods of time. This peculiar behaviour is then shown to lead to an analogous thermodynamic phase transition when varying T.

Let us consider the following measure over N intervals:

$$\mu_i(T) = \frac{|x(t+T) - x(t)|}{\sum_{t=1}^{N} |x(t+T) - x(t)|} \tag{1}$$

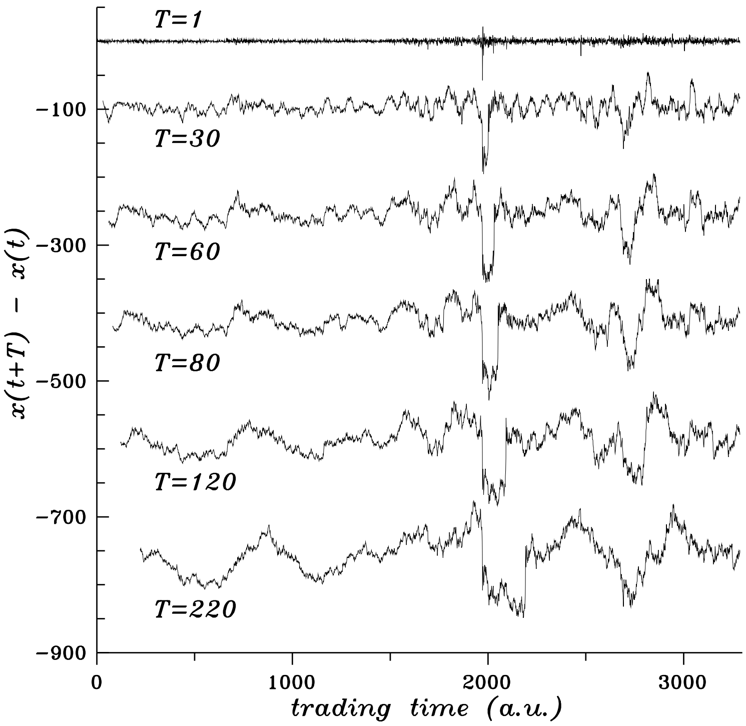

Figure 1. Temporal fluctuations of the S&P500 index for the period 1980–92 as a function of trading time lags $1 \leqslant T \leqslant 220$. The large, narrow fluctuations measured for $T = 1$ spread out on increasing T.

with T representing some finite trading time lags T. Clearly, the above relation can be viewed as a normalized probability measure with $\mu_i > 0$.

From μ_i in equation (1), we then construct the corresponding generating function Z, and its moments q, which follows the scaling

$$Z(q, N) = \lim_{T\to\infty} \sum_{t=1}^{N} \mu_i(T)^q \sim N^{-\tau(q)}. \tag{2}$$

To get a thermodynamic interpretation of multifractality (see, e.g., [19, 20, 25]), we divide the one-dimensional system of length L into N lines of length ℓ; thus $N \sim L/\ell$. We then associated this N with the number of discrete $x(t)$ time sequences considered in equation (1) in order to relate T, L and ℓ in the definition of the measure. For $\ell/L \to 0$, the function τ relates to the generalized fractal dimensions $(q-1)D_q$. Similar multifractal analysis has recently been performed for the energy dissipation field of turbulence [23], logistic maps [24] and surface roughening [25].

By following the thermodynamic formulation of multifractal measures, we can also derive an expression for the 'analogous' specific heat as follows:

$$C_q \equiv -\frac{\partial^2\tau(q)}{\partial q^2} \approx \tau(q+1) - 2\tau(q) + \tau(q-1). \tag{3}$$

For large T, we shall show that the form of C_q resembles a classical phase transition at a critical point.

We shall return to these equations in section 3.

3. Dynamical model

Similarly to [13, 26], in our approach there is only one stock. To study the price changes for one commodity it is necessary to derive a dynamical equation which results from the prevailing market conditions. The market is usually considered competitive so it self-organizes to determine the behaviour of prices. We assume here that all factors determining the demand D and the supply Q other than the asset price p remain constant over time and denote these quantities in equilibrium with an asterisk (*). In the following all variables are dimensionless.

In a competitive market it is expected that the rate of price increase should be a functional of the excess demand function $E(p) = D(p) - Q(p)$. Hence one writes $\mathrm{d}p/\mathrm{d}t \equiv f[E(p)]$ [27]. Assuming that a commodity can be stored then, in general, the flow of demand does not equal the flow of Q output. Hence stocks of the commodity (or product) build up when the flow of output exceeds the flow of demand and vice versa. Then the rate at which *the level of stocks S* changes can be approximated as $\mathrm{d}S/\mathrm{d}t = Q(p) - D(p)$. From these relations, a price adjustment relation that takes into account deviations of the stock level S above a certain optimal level S_o (to meet any demand reasonably quickly) is then given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\gamma \frac{\mathrm{d}S}{\mathrm{d}t} + \lambda(S_o - S) \tag{4}$$

where γ (i.e. the inverse of excess demand required to move prices by one unit [26]) and λ are positive parameters. If $\lambda = 0$, the price adjusts at a rate proportional to the rate at which stocks are either rising or running down. If $\lambda > 0$, prices would increase when stock levels are low and rise when they are high (with respect to S_o). Here we shall assume λ to characterize a noise term.

In our description, for each asset price p, we postulate simple nonlinear forms for the quantities D demanded and Q supplied such that

$$\begin{aligned} D(p) &= d^* + d_o \left[1 - \frac{\delta^2}{2!}(p - p^*)^2 + \cdots\right](p - p^*) \\ Q(p) &= q^* + q_o \left[1 - \frac{\delta^2}{2!}(p - p^*)^2 + \cdots\right](p - p^*) \end{aligned} \tag{5}$$

where d_o, q_o and $d^* = D(p^*)$, $q^* = Q(p^*)$ are arbitrary coefficients (related to material costs, wage rates, etc), p^* is an equilibrium price and δ is our order parameter. We write D and Q as a Taylor series expansion with the usual linear dependence (independent of δ) plus a nonlinear correction. Higher-order terms $\mathcal{O}(4)$ are here neglected for small $p - p^*$. In the above we might also consider two different δ, but to reduce variables to a minimum we assume D and Q to vary similarly from linearity. To simplify notation we also define

$$\beta_o \equiv q_o - d_o. \tag{6}$$

In the context of a simple economic model [27], it is reasonable to assume that S_o depends linearly on the demand; e.g. $S_o = \ell_o + \ell D$, with ℓ_o a constant and ℓ satisfying the constraint below. The postulated linear dependence of the optimal stock level S_o on D at equilibrium provides a complete economic model as in [27]. Therefore, in equilibrium (where $\frac{\mathrm{d}p}{\mathrm{d}t}|_{p^*} = 0$ and $\frac{\mathrm{d}S}{\mathrm{d}t}|_{S^*} = 0$, so that demand equals supply and $S = S_o$), from the above we obtain

$$d^* - q^* = 0 \qquad S^* = \ell_o + \ell(d^* + d_o p^*). \tag{7}$$

Following a small amount of algebra we find that the price of one strategic commodity is governed by the general equation

$$\frac{\mathrm{d}^2 p}{\mathrm{d}t^2} + (\gamma\beta_o - \ell\lambda d_o)\left[1 - \frac{3\delta^2}{2!}(p - p^*)^2\right]\frac{\mathrm{d}p}{\mathrm{d}t} + \lambda\beta_o(p - p^*)\left[1 - \frac{\delta^2}{2!}(p - p^*)^2\right] \approx 0. \tag{8}$$

The linear case is for $\delta = 0$. This leads to $p(t) - p^* \propto A_1 \cos(t\sqrt{\lambda\beta_o}) + A_2 \sin(t\sqrt{\lambda\beta_o})$.

3.1. The simple case $\ell\lambda d_o = \gamma\beta_o$

To keep the mathematics simple we choose $p^* = 0$ and consider ℓ to satisfy

$$\ell \equiv \frac{\gamma\beta_o}{\lambda d_o}. \tag{9}$$

The general equation (8) then reduces to

$$\frac{\mathrm{d}^2 p}{\mathrm{d}t^2} + \lambda\beta_o p - \frac{\delta^2\lambda\beta_o}{2} p^3 \approx 0. \tag{10}$$

This is our dimensionless price adjustment equation which gives rise to a burst as discussed later. When $\delta \neq 0$ and $[\lambda\beta_o, \delta^2\lambda\beta_o/2] > 0$, it has the well known kink solutions

$$p(t) = \pm\frac{\sqrt{2}}{\delta}\tanh\left(\sqrt{\frac{\lambda\beta_o}{2}}t\right). \tag{11}$$

Clearly, β_o of equation (6) must be positive. Since in a free market economy the demand for a product (or commodity) falls when its price increases, then it is reasonable to assume $d_o < 0$ in equation (5). As the price rises, the supply also increases; hence in general one also assumes $q_o > 0$. These conditions yield $\beta_o > 0$ as required by equation (11) and also $d_o\ell > 0$.

In the case $\delta < 0$ the above function for $p(t)$ displays a sudden decline around the equilibrium value p^* taken to be $p(t^*) = p(0) = 0$.

3.2. The case $\ell\lambda d_o \neq \gamma\beta_o$

If we consider the case in which $\ell\lambda d_o \neq \gamma\beta_o$, the price equation (8) results in a Lienard-type equation, $p'' + g_1(p)p' + g_o(p) = 0$, due to the presence of the $\frac{\mathrm{d}p}{\mathrm{d}t}$ term. What we present next is a brief discussion regarding its possible solutions. Again, we set $p^* = 0$.

With the aid of the substitution $\frac{\mathrm{d}p}{\mathrm{d}t} \equiv w(p)$, so that $\frac{\mathrm{d}^2p}{\mathrm{d}t^2} \equiv w(p)\frac{\mathrm{d}w}{\mathrm{d}p}$, the Lienard equation is reducible to an Abel equation of the second kind: $ww' = f_1(p)w + f_2(p)$. For small δ, the substitution $w(p) = p^3K(p) + \frac{p}{4}(\ell\lambda d_o - \gamma\beta_o)$ leads to the Bernoulli equation with respect to $p = p(K)$:

$$3K(p)\left[\frac{\delta^2}{8}(\ell\lambda d_o - \gamma\beta_o) - K(p)\right]\frac{\mathrm{d}p}{\mathrm{d}K} = K(p)p + \frac{(\ell\lambda d_o - \gamma\beta_o)}{4p} \tag{12}$$

whose solution is

$$p^2 = \left[\frac{\delta^2}{8}(\ell\lambda d_o - \gamma\beta_o) - K(p)\right]^{-1/3} \times \left\{K_o + \frac{(\gamma\beta_o - \ell\lambda d_o)}{8}\left(\frac{-1}{3K(p)}\right)^{2/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{\delta^2}{8}\frac{(\ell\lambda d_o - \gamma\beta_o)}{K(p)}\right)\right\} \tag{13}$$

with K_o an arbitrary constant and ${}_2F_1$ the first, complex hypergeometric function as arising in many physical problems. It converges within the unit circle $|(\ell\lambda d_o - \gamma\beta_o)/K(p)| < 2/\delta^2$.

By examining these equations one recognizes that if $\ell\lambda d_o \neq \gamma\beta_o$, other behaviour might appear for $p(t)$ (different from the one in equation (11)). However, it is important to note that such possible behaviour can essentially be found in the complex plane since the solutions of equation (13) are driven by the $K^{-2/3}$ term and depend on whether $\gamma\beta_o$ is greater or smaller than $\ell\lambda d_o$. A detailed analysis of such solutions is beyond the scope of this work. Here we only study the simplest dynamical economic model where all its variables, including prices, are related to the demand and supply functions as seen in a free market.

4. Model and analogous phase transition

Let us now identify the behaviour of $p(t)$ with an analogous phase transition as seen in multifractals. Following the analogy with critical phenomena in the time domain as proposed in [16], we also consider t to be the relevant variable for the analysis of the possible existence of an analogous critical point. To derive a connection between our economic model and multifractality we first briefly review multifractal phenomena in random multiplicative processes.

4.1. Multifractality in random multiplicative processes

Multifractality emerges in random multiplicative processes for a self-similar function ψ which is rescaled as $\hat{\psi}(x) = \mathrm{e}^{-L(1)x}\psi(x)$, where $L(1)$ is the generalized Lyapunov exponent for the first moment of ψ, and x is a space variable for N-disorder fluctuations on a unit interval [28]. All information about these systems is embodied in the non-zero, positive ψ measures.

To analyse the analogy of multifractality with thermodynamics one then scales the moments q of the functions $\hat{\psi}$ with respect to segments $l \leqslant 1$ as

$$Z(q,l) = \lim_{N\to\infty} \sum_{k=1}^{l} \hat{\psi}^{q}_{k,N} \sim l^{\tau(q)} \tag{14}$$

which defines the exponents τ and Z is a formal partition function.

It has been previously shown that for general random multiplicative processes [28], τ satisfies

$$\tau(q) = -(1-q) - \frac{1}{h}\{L(q) - qL(1)\} \tag{15}$$

where

$$\begin{aligned} L(q) &= \lim_{N\to\infty} \frac{1}{N} \ln \sum_{k=1}^{l} \psi^{q}_{k,N} - h \\ h &= \lim_{N\to\infty} \frac{1}{N} \ln l^{-1}. \end{aligned} \tag{16}$$

We shall use these findings to derive a connection between our economic model variables and multifractality phenomena. From this connection, we shall identify all the model quantities that might exhibit multifractality within the framework of a stochastic multiplicative process. These processes are known to generate power law probability density functions [29, 30].

4.2. General continuous probability theory

We next use the simple continuous probability theory discussed in [31] that allows one to explore the genesis of an 'analogous' phase transition from the point of view of a nonlinear singularity spectrum equivalent to multifractals. A crucial feature of this formalism is to consider $t\sqrt{\lambda\beta_o/2}$ to be a *continuous* random variable. Then, within the framework of general probability theory (see, e.g., [32]), the *distribution function* of this random variable, defined in a line and in terms of its *probability distribution* P, can be approximated as $\boldsymbol{P}\{\zeta_1 < \zeta' \leqslant \zeta_2\} = \mathcal{G}(\zeta_2) - \mathcal{G}(\zeta_1) \approx \int_{\zeta_1}^{\zeta_2} \phi(\zeta')\,\mathrm{d}\zeta'$, where {} indicates the function interval and ϕ is a uniform *probability density* that needs to be specified.

Inspired by well known results for the probability distribution function of real economic data with power-law tails [29, 30, 33–35], we assume $\phi(\zeta) \equiv (\phi_o/2)[1 - (\delta/\sqrt{2})H(\zeta)]$ such that $\phi(\zeta \to +\infty) \to 0$ and $\phi(\zeta \to -\infty) \to \phi_o > 0$, with H given by the real solutions of a static, dimensionless Ginzburg–Landau-like equation. It resembles the spin-flip function of the well known Glauber–Ising chain model. Using such solutions, we shall show next that it is possible to establish a relation with thermodynamics similarly to multifractality phenomena.

It is at this point that we relate $H(\zeta)$ to the positive solutions $p(t)$ of equation (10) and map $\zeta/\zeta_o \rightleftharpoons t\sqrt{\lambda\beta_o/2}$ within the framework of general probability theory. Following [31], it is then straightforward to derive an expression for an 'analogous' specific heat, C_ζ, for our economic system. To obtain such an expression we first evaluate the integral of $\boldsymbol{P}$ over the range $[\zeta_o, \zeta]$. Thus we have

$$\mathcal{G}(\zeta) - \mathcal{G}(\zeta_o) = \frac{\phi_o}{2}\int_{\zeta_o}^{\zeta}\left\{1 - \frac{\delta}{\sqrt{2}}\tanh\left(\frac{\zeta'}{\zeta_o}\right)\right\} \mathrm{d}\zeta' \equiv \tau(\zeta) \tag{17}$$

which, in turn, defines the function $\tau(\zeta)$. The $\mathcal{G}$ functions satisfy $\mathcal{G}(\zeta) > \mathcal{G}(\zeta_o)$, or alternatively, $\phi(\zeta) > \phi(\zeta_o)$ (since $\partial\mathcal{G}/\partial\zeta = \phi(\zeta)$ [32]), which is in accord with the above assumption for $\phi(\zeta \to \pm\infty)$.

Let us see next how our analysis from general probability theory would capture multifractality. Similarly to the dielectric breakdown model or the Poisson growth model, where the local field is set proportional to the growth probability [36], we assume here that the continuous function τ (to be identified as a free energy) is proportional to the probability distribution $\boldsymbol{P}$ as in equation (17).

After a little algebra, the above integral gives

$$\tau(\zeta/\zeta_o) \approx (1 - \zeta/\zeta_o)\tau(0) - \frac{\delta\zeta_o\phi_o}{2\sqrt{2}}\{\ln\cosh(\zeta/\zeta_o) - (\zeta/\zeta_o)\ln\cosh(1)\} \tag{18}$$

in which $\tau(0) \equiv \mathcal{G}(0) - \mathcal{G}(\zeta_o) = -\frac{\zeta_o\phi_o}{2}\{1 + \frac{\delta}{\sqrt{2}}\Gamma_\lambda\}$, such that $\Gamma_\lambda \equiv -\ln\cosh(1)$. We have assumed that $\delta\Gamma_\lambda/\sqrt{2} \ll 1$, hence $\delta < 0$.

Up to this point we have not carried out any actual numeration or scaling of a particular fractal configuration or set as used in multifractal theory. However, to gain further insight and explain how the present general probability can mimic multifractal behaviour, let us now relate the above τ to a measure within a random multiplicative process. By mapping our results to such a measure, all other quantities that can be scaling and are directly derived from τ, such as an analogous specific heat, will then follow.

From a comparison between equations (18) and (15) one can easily identify the following terms:

$$\begin{aligned} &\frac{1}{h} \rightleftharpoons \frac{\delta\zeta_o\psi_o}{2\sqrt{2}} \\ &\tau(0) \rightleftharpoons -1 \\ &L(\zeta/\zeta_o) \rightleftharpoons \ln\cosh(\zeta/\zeta_o). \end{aligned} \tag{19}$$

It is from this simple mapping between our results and those for random multiplicative processes that we can make an attempt to understand how the existence of the economic model order-parameter δ (and from it, the nonlinearities in the demand and supply functions) leads to obtaining multifractal-like behaviour.

From the above mapping we deduce that if $\delta \to 0$, then the moments q of the functions $\hat{\psi}$ with respect to segments l would vanish since $l \to 0$. It follows that a linear economic model would never exhibit multifractal features since in this case $\tau(q) = q - 1$. Furthermore, it is important to note that a Lyapunov exponent of the type we derive resembles that of a random multiplicative process with $\triangle$ described by the probability distribution $P(\triangle) = (1/n)\sum_{i=1}^{n}\delta(\triangle - \triangle_i)$ [28, 37]. By considering the actual definition of $L(q)$, the moments of the exponential measures in a random multiplicative process might well be related to our reduced variable as $q \rightleftharpoons \zeta/\zeta_o \rightleftharpoons t\sqrt{\lambda\beta_o/2}$.

4.3. Possible multifractal features

To analyse multifractal features in our economic model, when mapped into a random multiplicative process as discussed above, we consider standard definitions: $\tau(\zeta/\zeta_o) \equiv [(\zeta/\zeta_o)-1]D_\zeta$. According to such definitions [38–40], the function τ represents an 'analogous' free energy and D_ζ the multifractal dimension.

From equation (18) it follows that

$$D_\zeta \approx -\tau(0) + \frac{\delta\zeta_o\phi_o}{2\sqrt{2}(1-\zeta/\zeta_o)}\{\ln\cosh(\zeta/\zeta_o) - (\zeta/\zeta_o)\ln\cosh(1)\} \tag{20}$$

such that $\zeta \neq \zeta_o$. From this relation we obtain $D_{\zeta\to 0} = -\tau(0)$, and $D_{\zeta\to+\infty} = D_{\zeta\to 0} - \frac{\delta\zeta_o\phi_o}{2\sqrt{2}}\{1+\Gamma_\lambda\}$; $D_{\zeta\to-\infty} = D_{\zeta\to 0} + \frac{\delta\zeta_o\phi_o}{2\sqrt{2}}\{1-\Gamma_\lambda\}$. If $\zeta = \zeta_o$, then $D_{\zeta\to\zeta_o} = D_{\zeta\to 0} - \frac{\delta\zeta_o\phi_o}{2\sqrt{2}}\{\Gamma_\lambda + \tanh(1)\}$.

Complementary to τ we also define

$$\alpha(\zeta/\zeta_o) \equiv \frac{\partial}{\partial(\zeta/\zeta_o)}\tau(\zeta) \approx D_{\zeta\to 0} - \frac{\delta\zeta_o\phi_o}{2\sqrt{2}}\{\Gamma_\lambda + \tanh(\zeta/\zeta_o)\}. \tag{21}$$

It can be easily shown that $\alpha_{\max} \equiv \alpha(\zeta/\zeta_o \to -\infty) = D_{\zeta/\zeta_o\to-\infty}$, and $\alpha_{\min} \equiv \alpha(\zeta/\zeta_o \to +\infty) = D_{\zeta/\zeta_o\to+\infty}$.

Also according to multifractality phenomena, a possible analogy with thermodynamics can be established by relating τ to $f(\alpha) \equiv (\zeta/\zeta_o)\alpha(\zeta/\zeta_o) - \tau(\zeta/\zeta)$ via a Legendre transformation. From this analogy, where f is the analogous 'entropy', our analytical expression for the analogous 'specific heat' of the economic system becomes

$$C_\zeta \equiv -\frac{\partial^2\tau}{\partial(\zeta/\zeta_o)^2} \approx \frac{\delta\zeta_o\phi_o}{2\sqrt{2}}\operatorname{sech}^2(\zeta/\zeta_o). \tag{22}$$

5. Discussion

Let us see next how the type of behaviour given in figure 1 influences the 'analogous' specific heat function C_q defined in equation (3) and how it might characterize the onset of a crash for a real stock market index. In figure 2 we plot the 'analogous' specific heat C_q of the S&P500 index for four different time lags $T = 1, 10, 30, 80$. The + + + curve is for the 1984–8 data, × × × is for 1982–90 and o o o is for 1980–92.

For time lags $T \to 80$, we find that the main peak of our numerical C_q resembles a classical (first-order) physics phase transition at a critical point given by the main peak position. The peak turns symmetric around the value $q = -1$. Surprisingly, this 'analogous' specific heat C_q of the S&P500 index also displays a shoulder to the right of the main peak as a function of smaller time lags. Clearly, on decreasing T, the presence of the shoulder is a consequence of the large temporal $x(t+T)-x(t)$ fluctuations in this regime. Note that such peculiar behaviour for a double-peaked specific heat function is known to appear in the Hubbard model within the *weak*-to-*strong* coupling regime [41].

The relation in equation (2) requires $T \to \infty$ where the shoulder tends to vanish. It is this feature that make us believe that a large crash for a stock market index can be characterized by an 'analogous' specific heat which resembles a classical phase transition at a critical point as studied in multifractal physics.

We now turn to the results of our theoretical economic approach. For $T = 80$, the full curve in figure 2 represents theoretical results for C_ζ from equation (22) by choosing $\zeta \rightleftharpoons q+1$ to fit the main peak position. It can be seen that the theoretical C_ζ curve resembles the phase transition features of the real S&P500 economic data. Our simple model for one commodity sheds light on the main features observed regarding a possible analogous phase transition that

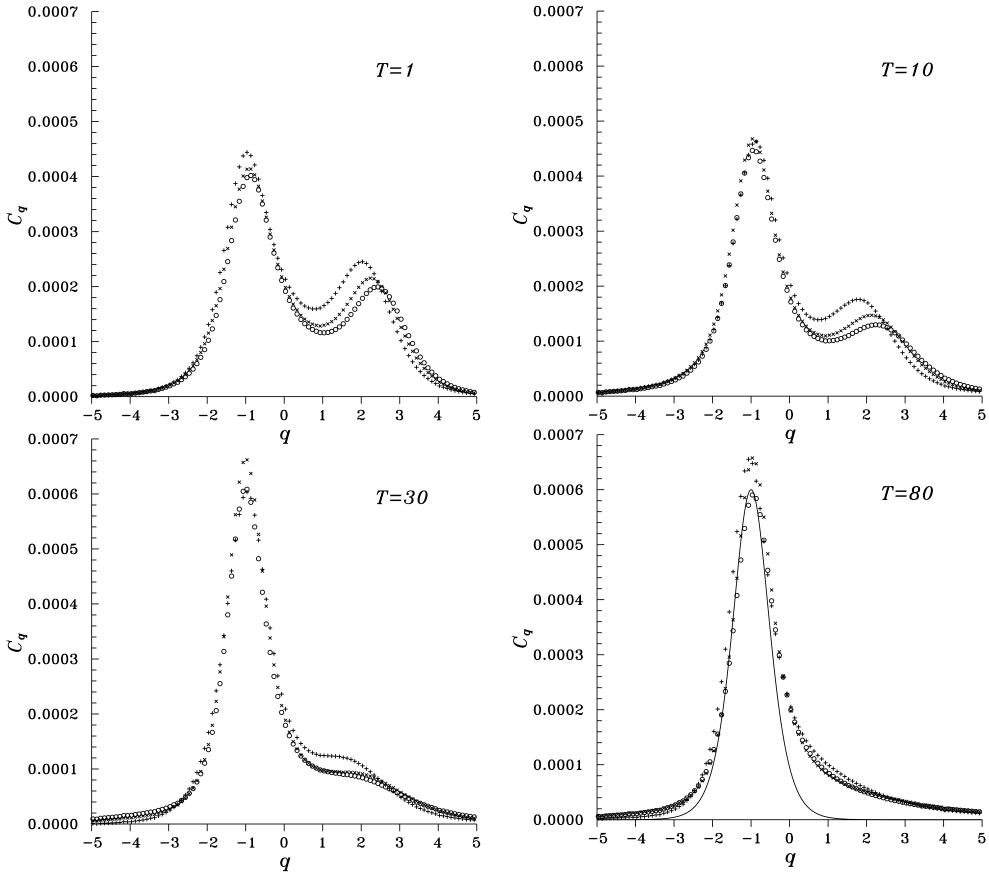

Figure 2. 'Analogous' specific heat C_q of the S&P500 index for different time lags $T = 1, 10, 30, 80$: The +++ curve is for the 1984–8 data set, × × × for 1982–90 and o o o for 1980–92. For $T = 80$, the full curve represents theoretical results for $C_{\zeta+1}$ from equation (22).

occurs when the excess demand becomes nonlinear (cf cubic p term in equation (10)) in terms of the price for one commodity. We believe the width difference between both curves, i.e. the analytical C_ζ and the estimated $C(q)$ function from historical S&P500 data, is due to the fact that the S&P500 price index is made of large-capitalization stocks representing a 'basket' or portfolio of commodities.

For large time lags, there is a sharp peak that resembles the quantitative signals measured in multifractals [38–40]. From this feature we presume the existence of an analogous, say, critical point ζ^* above which inflated prices for one strategic commodity might be found. The maximum and minimum values of α in equation (21) (for more details also see [31]) allow for the existence of a critical point ζ^* above which the infinite hierarchy of phases can be found, but below which a single phase appears characterized by $\alpha_{\max}$. It resembles a classical phase transition at a critical point [18].

Of course, the analogy between multifractality and a thermodynamic phase transition as discussed here does not imply that the economic system has a phase transition. What we have shown, as a direct consequence of the p^3 term in equation (10), is that prices can became inflated prior to equilibrium (i.e. $t < 0$ by convention), whereas after a sudden crash prices

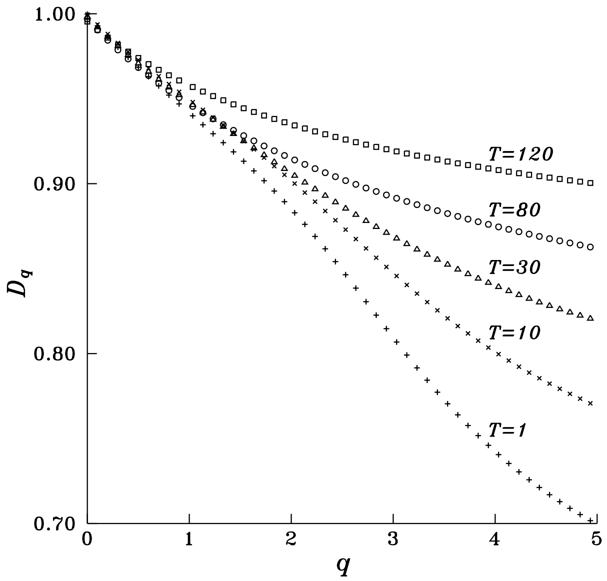

Figure 3. Dimension spectrum D_q of the S&P500 index for the 1980–92 period in the range $q \geqslant 0$ for different time lags $T = 1, 10, 30, 80, 120$.

might devalue. The greater $|\delta|$ values are taken, the smaller the price reduction becomes. If $p > 0$ the amount supplied exceeds demand and stocks accumulate, whereas if $p < 0$ the stocks deplete. In addition, by tuning $\delta \to 0$ in equation (11), we are able to predict that prices can decrease (or increase if $A_2 > 0$) monotonically with $C \to 0$ for all ζ.

The hill behaviour of the generalized dimensions D_q for $q < 0$ is a well known fact when using the box-counting method as in this work [42, 43]. Thus, we check if D_q is sufficiently smooth for C_q to be meaningful. In figure 3 we represent D_q of the S&P500 index for the 1980–92 period within the range $q \geqslant 0$ for the time lags $T = 1, 10, 30, 80, 120$. The total number of sequence points analysed include 1495 points for the different C_q curves, 1500 points for D_q and 9864 points for the S&P500 1980–92 data set. We also fit these non-seasonal data by standard exponential smoothing techniques, and from the fitting our error estimates for the D_q curves is found to be less than 3%. Our results for D_q at negative q (not shown) follow a typical convergent behaviour as can be seen, for example, in [31].

From figure 3 it can be seen that, when increasing $T > 30$, D_q is fully multifractal-like and for $T > 120$ it becomes flatter (i.e. uniform measure) and tends to one independently of q, so multifractality becomes smaller. For lower values of $T < 30$ we find a non-monotonous decreasing behaviour of D_q, conceivable within the double-peaked form of C_q displayed in figure 2, which relates to the presence of the onset of crash for the S&P500 stock index in figure 1. Using this data set, we also estimated the multifractality strength of the time sequence by considering the limit $1 - D_{q\to\infty}$ for the few different time lags displayed in figure 3. We find that that this quantity does not follow a power-law scaling for T as in [43] for values $T \to 1$. This is a consequence of the complex network of trading interactions comprised in our nonlinear forms for the supply and demand functions.

The present choice for a excess demand function of the form $E(p) \propto (p - p^*) - \frac{\delta^2}{2}(p - p^*)^3$, with $\delta \neq 0$ plays a key role in our description. We have $d^* = q^*$, hence the second price derivative $\frac{\mathrm{d}^2 E}{\mathrm{d}p^2} \approx 3\beta_o\delta^2 p$ (or that of D and Q) is price dependent. It is such a behaviour of E (independently of the sign of δ), that leads to an abrupt fluctuation in the price dynamics and characterizes multifractality phenomena.

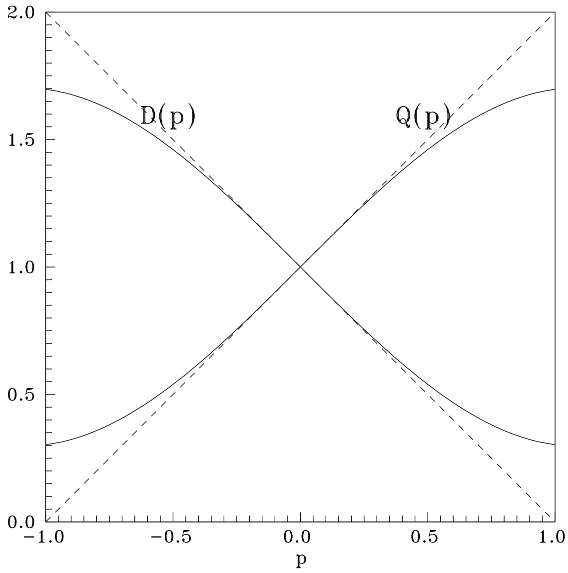

Figure 4. Plots of the linear p-dependence of D and Q independent of δ (dotted lines) and the assumed nonlinear forms for the demand and supply of equation (5) with $|\delta p| \ll 1$ (full curves).

Our expressions for the demand and supply functions of equation (5) are justified as follows. As seen in figure 4, the (commonly used) linear p-dependence for D and Q and our assumed nonlinear form for the these functions display similar behaviour when $|\delta p| \ll 1$. Even more important, this figure depicts the fact that as price falls, the quantity demanded for a commodity can increase in agreement with one of the basic principles of economy. Since the demand curve can indeed change in a number of ways which may not be at all obvious, similar tails to our D and Q functions has also been previously hypothesized in [44].

In the real world, exceptions to the general law of demand can take place causing the D curve to increase upwards from low to high prices. However, these exceptions, which include goods of conspicuous consumption—as, for example, certain articles of jewellery—are not very important [45]. Theoretically, this would simply mean to set $d_o > 0$ in our $D(p)$ function independently of the order parameter δ. The upward dependence of D can occur as soon as there is speculation (as in the precious stones mentioned, but also for market prices). The idea is that when price increases, investors buy, because they hope the price will keep climbing. This is called a 'trend-following' investment strategy. On the other hand, our choice for Q (with $q_o > 0$) also follows the typical behaviour observed in a competitive market (where no individual producer can set his own desired price): that is, the higher the price, the higher the profit, then the higher the supply.

It is also important to mention that the present choice for D and Q, and their linear p-dependence both lead to the same linear relation for D versus Q, namely $(D - d^*)/d_o = (Q - q^*)/q_o$. Such a linear behaviour finds frequent use in applied economics.

Furthermore, it is well known in econometrics that demand functions are somewhat abstract quantities since all of these data are taken to refer to possible events at just one moment of time [46]. In particular, consumer behaviour (i.e. tastes, desires, etc) can shift a (concave or convex) D curve (which may resemble the tails of our D and Q functions). Also, typical demand and supply of capital look like a step function [47].

Usually, additional hypothetical information is needed to make up realistic demand relationships (e.g., consumer interviews). In principle, one might also determinate D out of many data sources from different economic sectors using standard statistical techniques for multiple regression time series analysis. But the effectiveness of such an approach varies case by case. In view of all of this, our nonlinear approach for D and Q may be well placed for simulating market situations leading to a sudden decline around equilibrium for the price of a commodity. To this end, we add that a commodity price function displaying an inflection point at a characteristic frequency has also been theoretically discussed in [48].

Another important test for our D and Q expressions arises by considering possible aggregated changes in conditions of demand (or supply). Because of such aggregated changes, e.g. due to buyers' income and scale of preferences, it is always difficult to estimate how much of the change in D is due to price alone. To know what these effects are upon D, and obtain the degree of responsiveness of demand to price variations, it is necessary to study *the point price elasticity of demand* curves, defined as $\varepsilon \equiv -\mathrm{d}\log D/\mathrm{d}\log p$ [46]. This quantity is said to be elastic (or flat) if $\varepsilon > 1$, or inelastic if $\varepsilon < 1$ depending on the demand schedule.

By assuming a linearly decreasing p-dependence for D one obtains the expression $\varepsilon = -\frac{1}{1+(d^*/d_o p)}$. And if $\delta \neq 0$ one gets $\varepsilon \approx -\frac{1-3(\delta p)^2/2}{[1-(\delta p)^2/2]+(d^*/d_o p)}$. Hence we find that a transition from an inelastic to an elastic demand curve appears at $p \approx -\frac{d^*/d_o}{1-(\delta p/2)^2}$, whereas by considering the common linear dependence of $D(p)$ one finds $p \approx -d^*/d_o$. Therefore, both approaches lead to similar results for the elasticity of demand (or supply) if $\delta \to 0$.

6. Concluding remarks

We have found that within the framework of multifractal physics, the 'analogous' specific heat of the S&P500 discrete price index displays a shoulder to the right of the main peak for low time-lag values. On decreasing T, the presence of the shoulder is a consequence of the peaked, temporal $x(t+T) - x(t)$ fluctuations in this regime. For large time lags ($T > 80$), we have found that C_q displays typical features of a classical phase transition at a critical point according to multifractal physics.

Our simple continuous model for one commodity mimics the main observed features of $C(q)$ for large trading time lags. We believe the width difference between the analytical C_ζ and estimated $C(q)$ curves from historical S&P500 data is due to the fact that the S&P500 index comprises many commodities. From these results we conclude that an analogous phase transition might occur when the excess demand becomes nonlinear (cf cubic p term in equation (10)) in terms of the commodity price.

We have assumed that there is only one stock in which a commodity can be stored. The market has been considered competitive so it self-organizes to determine the behaviour of prices. All factors determining D and Q other than p are assumed to remain constant over time. There exists a price adjustment relation that takes into account deviations of the stock level S above a certain optimal level S_o characterized by a noisy λ parameter. We have postulated simple nonlinear forms for the quantities D demanded and Q supplied, with δ the order parameter, and have neglected higher-order terms in their expansion on $p - p^*$. We have followed the context of other simple economic models and assumed that the optimal stock level S_o depends linearly on the demand (with ℓ being the slope). We have considered ℓ to satisfy the constraint in equation (9), relating the economic model variables: γ, λ, q_o and d_o. We have $d^* = q^*$ and set the equilibrium price p^* to 0. We have identified the behaviour of $p(t)$ with an analogous phase transition as seen in multifractals, by considering t to be the relevant variable and using a simple continuous probability theory.

We have related $t\sqrt{\lambda\beta_o/2}$ to a *continuous* random variable and have related its probability distribution function to our solutions for $p(t)$ given in equation (11). Our definitions for the analogous thermodynamic variables have been made according to definitions used in multifractal physics and by mapping to multifractality phenomena in random multiplicative processes.

Of course, the scenario of a transition from, say, inflated to devalued price changes in the time domain is pure speculation. However, a great deal of relevant information has been extracted from the present continuous approach which, essentially, does rely on δ only. Our description presumes the existence of a stationary probability function $\boldsymbol{P}$ as in equation (17). We have assumed $\boldsymbol{P}$ to be proportional to the continuous function τ (identified as a free energy) similarly to the dielectric breakdown model or the Poisson fractal growth models [36]. We add that this type of approximation is also used when modelling earthquakes, where the probability function of the total number of relaxations (size) of the earthquakes is set proportional to the energy release during an earthquake [49].

Within the framework of general probability theory (see, e.g., [32]), a continuous arbitrary random variable can have an associated probability distribution (if it is discrete) or a positive probability density (if it is continuously distributed). The later is required to be differentiable. These can then be related by an integral equation of the type used in this work to then derive the analogous thermodynamics equation. To do this we have proceed as follows. We have used the characterization of multifractal singularities—where an analogy with thermodynamics has been established in the literature—to derive, and associate a meaning to our analogous quantities. The main point to understand is that we have not carried out any direct numeration or scaling of particular fractal configurations or sets, but we have evaluated the integral of the probability distribution $\boldsymbol{P}$ (for the continuous random variable ζ) assumed to be related with our $p(t)$ function and mapped the results for τ to those of a random multiplicative process. Our equivalent definitions for the thermodynamic variables, derived by solving such an integral, followed from the convention used in multifractal physics in the sense that τ must not be a linear function of ζ (see, e.g., [38]). Alternative approaches can also be found in [50, 51]. In all these cases, analogous thermodynamic quantities follow from the Legendre transform of $\tau(\zeta)$.

As discussed in [31, 38], the concept of a phase transition in multifractal spectra was first found in the study of logistic maps, Julia sets and other simple systems. Evidence was then found for a phase transition in more complex random systems such as diffusion-limited aggregation. The condition for an analogy between multifractality and a thermodynamic phase transition is that the analogous free energy (α in our notation) undergoes a quite sharp jump near a critical value ζ^*. For values of $\zeta < \zeta^*$, the analogous free energy τ is dominated by the maximum energy term $\alpha_{\max}$ and a singular behaviour of the specific heat is found at this point. A well-defined analogous entropy function $f(\alpha)$ also suggests the existence of an analogous phase transition. We have shown that the present kinetic description satisfies such properties obeying the constraints for ℓ in equation (9). This condition simplifies the analysis and corresponds to the case in which the term for the first derivative of p with respect to t (i.e. $\frac{\mathrm{d}p}{\mathrm{d}t}$) is absent in the price adjustment equation (10).

Application of multifractal analysis to discrete one-dimensional time sequences as derived, for example, from the cellular automaton model of a rice-pile model is not new (see, e.g., [43]). Nevertheless, to our best knowledge, we have related multifractal physics to financial time series for the first time. Our main contribution has been the analysis of the 'analogous' specific heat (or second derivatives) C_q of the data sequence, in conjunction with the analytical form derived from our proposed one-stock model, which suggest typical features of a classical physics phase transition at a critical point. The double-peaked form of C_q is a consequence of the presence of the onset of crash for the S&P500 stock index.

Our work also differs from previous formalisms of multifractality of time series in that we have analytically characterized multifractal singularities and given a thermodynamics interpretation. The suggested nonlinear analytical forms for the supply and demand functions of a commodity lead us to derive theoretically the observed features of a classical phase transition using the simplest economic model. We believe that these novel aspects of the topic could stimulate further investigations in this direction and might be important to open beneficial discussions in the field of econophysics.

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